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GAS-DYNAMIC ANALOGY FOR VORTEX FREE-BOUNDARY FLOWS

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The classical shallow-water equations describing the propagation of long waves in flow without a shear of the horizontal velocity along the vertical coincide with the equations describing the isentropic motion of a polytropic gas for a polytropic exponent $\gamma = 2$ (in the theory of fluid wave motion, this fact is called the gas-dynamic analogy). A new mathematical model of long-wave theory is derived that describes shear free-boundary fluid flows. It is shown that in the case of one-dimensional motion, the equations of the new model coincide with the equations describing nonisentropic gas motion with a special choice of the equation of state, and in the multidimensional case, the new system of long-wave equations differs significantly from the gas motion model. In the general case, it is established that the system of equations derived is a hyperbolic system. The velocities of propagation of wave perturbations are found.

Key words: *long-wave approximation, shear flow, free boundary, shallow water, gas-dynamic analogy.*

1. Averaging of the Long-Wave Equations. The motion of an ideal incompressible free-boundary fluid is described by Euler's equations

$$\rho \frac{d\boldsymbol{u}}{dt} + \nabla_2 p = 0, \qquad \rho \frac{du_3}{dt} + p_{x_3} = -\rho g, \qquad \operatorname{div}_2 \boldsymbol{u} + u_{3x_3} = 0, \tag{1.1}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla_2 + u_3 \frac{\partial}{\partial x_3}.$$

The free boundary $x_3 = h(t, x_1, x_2)$ is subjected to the kinematic and dynamic boundary conditions

$$h_t + (\boldsymbol{u}_2 \cdot \nabla_2) h = u_3, \qquad p = p_0 = \text{const}, \tag{1.2}$$

and the even bottom $x_3 = 0$ to the nonpenetration condition

$$u_3 = 0. \tag{1.3}$$

In (1.1)–(1.3), t is time, $\boldsymbol{x} = (x_1, x_2)$ is the radius-vector in the horizontal plane, x_3 is the vertical coordinate, $\boldsymbol{u} = (u_1, u_2)$ is the horizontal velocity, u_3 is the vertical fluid-velocity component, ρ is the density, $h(t, x_1, x_2)$ is the depth, p is the pressure, g is the acceleration of gravity, and ∇_2 and div_2 are the gradient and divergence calculated with respect to the vector variable $\boldsymbol{x} = (x_1, x_2)$.

We introduce the dimensionless variables

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{L}, \quad x_3' = \frac{x_3}{H}, \quad t' = \frac{Ut}{L}, \quad \boldsymbol{u}' = \frac{\boldsymbol{u}}{U},$$

$$u_3' = \frac{Lu_3}{UH}, \quad h' = \frac{h}{H}, \quad p' = \frac{p}{RU^2}, \quad \rho' = \frac{\rho}{R}.$$

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In these variables, the Euler equation (1.1) are written as

$$\rho\,\frac{d\boldsymbol{u}}{dt}+\nabla_2 p=0,\qquad \varepsilon^2\rho\,\frac{du_3}{dt}+p_{x_3}=-\rho\,\mathrm{Fr}^{-2},\qquad \operatorname{div}_2\boldsymbol{u}+u_{3x_3}=0 \tag{1.4}$$

(the primes in the notation of the new dimensionless variables are omitted; $\mathrm{Fr}=U/\sqrt{gH}$ is the Froude number and $\varepsilon=H/L$). Eliminating the pressure p from system (1.4), we obtain the Helmholtz equation which describes the evolution of the dimensionless vortex $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)=(\varepsilon^2u_{3x_2}-u_{2x_3},u_{1x_3}-\varepsilon^2u_{3x_1},u_{2x_1}-u_{1x_2})$:

$$\boldsymbol{\omega}_t+(\boldsymbol{u}\cdot\nabla_2)\boldsymbol{\omega}+u_3\boldsymbol{\omega}_{x_3}=(\varepsilon^2u_{3x_2}-u_{2x_3})\boldsymbol{U}_{x_1}+(u_{1x_3}-\varepsilon^2u_{3x_1})\boldsymbol{U}_{x_2}+(u_{2x_1}-u_{1x_2})\boldsymbol{U}_{x_3} \tag{1.5}$$

$[\boldsymbol{U}=(u_1,u_2,u_3)]$. Projecting the Helmholtz equation (1.5) onto the x_1 and x_2 axes, we obtain the equations with a small right-hand side of order $O(\varepsilon^2)$:

$$\begin{aligned}&(u_{1x_3})_t+(\boldsymbol{u}\cdot\nabla_2)u_{1x_3}+u_3(u_{1x_3})_{x_3}+u_{1x_2}u_{2x_3}-u_{2x_2}u_{1x_3}=O(\varepsilon^2),\\&(u_{2x_3})_t+(\boldsymbol{u}\cdot\nabla_2)u_{2x_3}+u_3(u_{2x_3})_{x_3}+u_{2x_1}u_{1x_3}-u_{1x_1}u_{2x_3}=O(\varepsilon^2).\end{aligned} \tag{1.6}$$

In the derivation of the long-wave model, the terms of order $O(\varepsilon^2)$ in Eqs. (1.6) can be ignored. Then, the equation for the vertical momentum component reduces to the hydrostatic law of pressure distribution with depth:

$$p_{x_3}=-\rho\,\mathrm{Fr}^{-2},\qquad p-p_0=\rho\,\mathrm{Fr}^{-2}(h-x_3).$$

Using this representation, we obtain the following approximate equations of the model of long waves propagating in shear flow:

$$\frac{d\boldsymbol{u}}{dt}+\mathrm{Fr}^{-2}\,\nabla_2 h=0,\qquad \operatorname{div}_2\boldsymbol{u}+u_{3x_3}=0,\qquad h_t+(\boldsymbol{u}^h\cdot\nabla_3)h=u_3^h. \tag{1.7}$$

Here $\boldsymbol{u}^h$ and u_3^h are the velocity components on the free boundary $x_3=h(t,x_1,x_2)$. At the bottom $x_3=0$, the solution of system (1.7) should satisfy condition (1.3).

Solutions of system (1.4) that satisfy the condition $S=(u_{1x_3})^2+(u_{2x_3})^2\neq 0$ will be called flows with a vertical velocity shear or shear flows. Accordingly, in a shearless flow, $u_{1x_3}=u_{2x_3}=0$. Model (1.7) reduces to a system of integrodifferential equations to which the theory of generalized characteristics is applicable (see [1, 2]), which allows one to study the general properties of long waves propagating in shear flow. Below, simpler models will be derived in which the shear nature of the flow is taken into account by introducing some average characteristics of the velocity shear along the vertical.

In the class of flows with a fairly small quantity S, the classical shallow-water equations can be derived by averaging Eqs. (1.7) over the depth. Integrating (1.7) over x_3 from 0 to h and taking into account the boundary conditions, we obtain

$$\Big(\int_0^h\boldsymbol{u}\,dx_3\Big)_t+\operatorname{div}\Big(\int_0^h(\boldsymbol{u}\otimes\boldsymbol{u})\,dx_3\Big)+\frac{\mathrm{Fr}^{-2}}{2}\nabla_2(h^2)=0, \tag{1.8}$$

$$h_t+\operatorname{div}\Big(\int_0^h\boldsymbol{u}\,dx_3\Big)=0,$$

where $(\boldsymbol{a}\otimes\boldsymbol{b})$ is a dyad of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Introducing the horizontal velocity averaged over the depth

$$\bar{\boldsymbol{u}}=h^{-1}\int_0^h\boldsymbol{u}\,dx_3,$$

we replace the integral of the vector $\boldsymbol{u}$ over the depth in Eqs. (1.8) by the expression $h\bar{\boldsymbol{u}}$. However, the integrals of the expressions quadratic in velocity in Eqs. (1.8) are not expressed in terms of the averaged velocity in the general shear flow. In hydraulics, these integrals are approximated by empirical formulas of the form [3]

$$\int_0^h(\boldsymbol{u}\otimes\boldsymbol{u})\,dx_3=\alpha h(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}),$$

where α is an empirical correction factor. We note that the use of this relation leads to a loss of invariance of the resulting system of equations with respect to the Galilean transformation, notwithstanding the fact that system (1.8) admits this transformation.

In [4], system (1.8) for the one-dimensional case was closed by a different empirical relation:

$$\frac{\partial}{\partial x_1}\int_0^h (u-\bar u)^2\,dx_3 = gH\,\frac{\partial h}{\partial x_1}$$

(H is a certain constant). However, a justification of this relation was not given.

A model which on the average approximately accounts for the shear nature of the flow is obtained below without invoking empirical formulas. Using the obvious identity $\boldsymbol{u} = \bar{\boldsymbol{u}} + (\boldsymbol{u} - \bar{\boldsymbol{u}})$, we calculate the tensor $\boldsymbol{u}\otimes\boldsymbol{u}$ in the form

$$\boldsymbol{u}\otimes\boldsymbol{u} = \bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}\otimes(\boldsymbol{u}-\bar{\boldsymbol{u}}) + (\boldsymbol{u}-\bar{\boldsymbol{u}})\otimes\bar{\boldsymbol{u}} + (\boldsymbol{u}-\bar{\boldsymbol{u}})\otimes(\boldsymbol{u}-\bar{\boldsymbol{u}}).$$

In the integration of this expression from 0 to h, the terms linear in $(\boldsymbol{u}-\bar{\boldsymbol{u}})$ make zero contribution. As a result, we obtain the representation

$$\int_0^h (\boldsymbol{u}\otimes\boldsymbol{u})\,dx_3 = h(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + P,$$

where

$$P = \int_0^h (\boldsymbol{u}-\bar{\boldsymbol{u}})\otimes(\boldsymbol{u}-\bar{\boldsymbol{u}})\,dx_3.$$

Using the tensor P, we write Eqs. (1.8) as

$$\begin{gathered} h_t + \operatorname{div}_2(h\bar{\boldsymbol{u}}) = 0, \\ (h\bar{\boldsymbol{u}})_t + \operatorname{div}_2(h(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + P) + (1/2)\,\mathrm{Fr}^{-2}\,\nabla_2(h^2) = 0. \end{gathered} \tag{1.9}$$

The above system of equations is not closed because, along with h, $\bar{\boldsymbol{u}}$, it contains the unknown components of the tensor P:

$$P_{ij} = \int_0^h (u_i-\bar u_i)(u_j-\bar u_j)\,dx_3.$$

We note that Eqs. (1.8) have particular solutions of the form

$$u_1 = \omega y + u_0(t,x), \quad u_2 = 0, \quad h = h(t,x) \qquad (\omega = \mathrm{const}).$$

Calculating the component P_{11} on this solution, we obtain

$$P_{11} = \omega^2 h^3/12. \tag{1.10}$$

Generally, the components P_{ij} are not expressed in terms of the other sought functions. We obtain the equations governing the evolution of these components. Averaging Eqs. (1.8) over the depth

$$\frac{d}{dt}(u_iu_j) + \mathrm{Fr}^{-2}(h_{x_i}u_j + h_{x_j}u_i) = 0,$$

we have

$$\frac{\partial}{\partial t}(h\bar u_i\bar u_j + P_{ij}) + \sum_{k=1}^{2}\frac{\partial}{\partial x_k}(h\bar u_i\bar u_j\bar u_k + \bar u_kP_{ij} + \bar u_iP_{jk} + \bar u_jP_{ik} + P_{ijk})\,\mathrm{Fr}^{-2}\,h(\bar u_jh_{x_i} + \bar u_ih_{x_j}) = 0.$$

Here

$$P_{ijk} = \int_0^h (u_i - \bar{u}_i)(u_j - \bar{u}_j)(u_k - \bar{u}_k)\, dx_3$$

are the components of the new unknown third-rank tensor which are expressed in terms of the third moments of the differences of the true and averaged velocities $(\boldsymbol{u} - \bar{\boldsymbol{u}})$ (analogs of the pulsation velocities in turbulent flow theory). Complementary equations describing the evolution of P_{ijk} are required to close the system of equations obtained. Calculations similar to those performed above show that these complementary equations contain integrals that depend on the fourth-order moments of the pulsation velocities. Continuing this process infinitely, we obtain a system with an infinite number of equations and unknown functions. This situation is characteristic of the cases of averaging of nonlinear equations (a similar situation arises in constructing the closed system of the equations for the moments of pulsation velocities in turbulence theory). Thus, in the general shear flow, the averaged equations are not closed in a finite number of steps. The description of motion in terms of average quantities reduces to solving an infinite system of differential equations. Below, it is shown that in the class of flows with weak shear, closure can be implemented within the framework of approximate theory.

From Eqs. (1.6), it follows that if at $t = 0$

$$u_{1x_3} = O(\varepsilon^{\alpha}), \qquad u_{2x_3} = O(\varepsilon^{\alpha}),$$

then for all $t > 0$,

$$u_{1x_3} = O(\varepsilon^{\beta}), \qquad u_{2x_3} = O(\varepsilon^{\beta})$$

$[\beta = \min(2, \alpha)]$.

If $\beta \geq 1$, then $|S| \ll \varepsilon^2$ and the motion of a free-boundary fluid can be described by the classical shallow-water equations

$$h_t + \mathrm{div}_2(h\bar{\boldsymbol{u}}) = 0,$$

$$(h\bar{\boldsymbol{u}})_t + \mathrm{div}_2(h(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}})) + (1/2)\,\mathrm{Fr}^{-2}\,\nabla_2(h^2) = 0.$$

Indeed, in Eqs. (1.8), the tensor P, which has the order of smallness $\varepsilon^{2\beta}$ can be omitted (taking into account that $\varepsilon^{2\beta} \ll \varepsilon^2$) because

$$|\boldsymbol{u} - \bar{\boldsymbol{u}}| = O(\varepsilon^{\beta}), \qquad |P_{ij}| = O(\varepsilon^{2\beta}), \qquad |P_{ijk}| = O(\varepsilon^{3\beta}).$$

In this case, the error is smaller than a quantity of order $O(\varepsilon^2)$.

We note that the classical system of shallow-water equations coincides, to within the renotation of the required functions, with the equations describing the isentropic motion of a polytropic gas for $\gamma = 2$. In this case, the equation of state for the "gas" is written as follows [5, 6]:

$$p(h) = \mathrm{Fr}^{-2}\, h^2/2.$$

If $\beta < 1$, then $|P_{ij}| = O(\varepsilon^{2\beta}) \gg O(\varepsilon^2)$ and, notwithstanding that the terms P_{ij} are small for small ε, their value far exceeds the error of the derivation of the long-wave approximation model (1.8). Hence, in the construction of the approximate theory taking into account the small quantities $O(\varepsilon^{2\beta}) \gg O(\varepsilon^2)$, the terms P_{ij} should also be taken into account. In this case, the terms P_{ijk} of order $O(\varepsilon^{3\beta})$ can be omitted since $\varepsilon^{3\beta} \ll \varepsilon^{2\beta}$ for $\beta < 1$.

Below, we assume that the solution of Eqs. (1.6) describes motion with a weak shear if

$$u_{1x_3} = O(\varepsilon^{\beta}), \qquad u_{2x_3} = O(\varepsilon^{\beta})$$

for $\beta < 1$.

In the class of weak shear flows, we can close system (1.8), (1.9) by omitting the third-order moments in Eqs. (1.9). As a result, we have the following equations for h, $\bar{u}$, and P_{ij}:

$$h_t + \mathrm{div}_2(h\bar{\boldsymbol{u}}) = 0,$$

$$(h\bar{\boldsymbol{u}})_t + \mathrm{div}_2(h(\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) + P) + (1/2)\,\mathrm{Fr}^{-2}\,\nabla_2(h^2) = 0; \tag{1.11}$$

$$(P_{ij})_t + \operatorname{div}(P_{ij}\bar{\boldsymbol{u}}) + \sum_{k=1}^{2}\Big(P_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + P_{ik}\frac{\partial \bar{u}_j}{\partial x_k}\Big) = 0. \tag{1.12}$$

Thus, we obtained a new mathematical model which on the average accounts for the effect of the weak shear of the horizontal velocity along the vertical and generalizes the classical shallow-water model.

Instead of the variables P_{ij}, we introduce the variables Q_{ij} related to them by the formula

$$P_{ij} = h^3 Q_{ij}/12$$

written by analogy with (1.10). In the new variables, Eqs. (1.11) and (1.12) have the form

$$h_t + \operatorname{div}_2(h\bar{\boldsymbol{u}}) = 0,$$

$$\frac{d\bar{u}_1}{dt} + \frac{h^2}{12}\frac{\partial Q_{11}}{\partial x_1} + \frac{h^2}{12}\frac{\partial Q_{12}}{\partial x_2} + \Big(\frac{hQ_{11}}{4} + \mathrm{Fr}^{-2}\Big)\frac{\partial h}{\partial x_1} + \frac{hQ_{12}}{4}\frac{\partial h}{\partial x_2} = 0,$$

$$\frac{d\bar{u}_2}{dt} + \frac{h^2}{12}\frac{\partial Q_{12}}{\partial x_1} + \frac{h^2}{12}\frac{\partial Q_{22}}{\partial x_2} + \frac{hQ_{12}}{4}\frac{\partial h}{\partial x_1} + \Big(\frac{hQ_{22}}{4} + \mathrm{Fr}^{-2}\Big)\frac{\partial h}{\partial x_2} = 0,$$

$$\frac{dQ_{11}}{dt} + 2Q_{12}\frac{\partial \bar{u}_1}{\partial x_2} - 2Q_{11}\frac{\partial \bar{u}_2}{\partial x_2} = 0, \tag{1.13}$$

$$\frac{dQ_{12}}{dt} - Q_{12}\operatorname{div}_2(\bar{\boldsymbol{u}}) + Q_{11}\frac{\partial \bar{u}_2}{\partial x_1} + Q_{22}\frac{\partial \bar{u}_1}{\partial x_2} = 0,$$

$$\frac{dQ_{22}}{dt} + 2Q_{12}\frac{\partial \bar{u}_2}{\partial x_1} - 2Q_{22}\frac{\partial \bar{u}_1}{\partial x_1} = 0.$$

From the definition of the quantities P_{ij} and the Cauchy inequalities, it follows that

$$P_{11}P_{22} \geq P_{12}^2.$$

We note that the components Q_{ij} are related by the similar inequality

$$Q_{11}Q_{22} \geq Q_{12}^2. \tag{1.14}$$

From system (1.13), for the quantity $J = Q_{11}Q_{22} - Q_{12}^2$ we obtain the equation

$$\frac{dJ}{dt} - 2J\operatorname{div}_2\boldsymbol{u} = 0,$$

which implies that if $J = 0$ (or $J \geq 0$) at $t = 0$, then $J = 0$ ($J \geq 0$) for all values of t. We also note that at $t = 0$, J vanishes if the initial values of the horizontal velocity components depend linearly on the vertical variable x_3, i.e., if the equalities $u_{1x_3x_3=0}$ and $u_{2x_3x_3=0}$ are satisfied at $t = 0$.

2. Hyperbolicity of the Long-Wave Equations. To find the characteristics of system (1.13), we write it in vector form

$$\boldsymbol{U}_t + A\boldsymbol{U}_{x_1} + B\boldsymbol{U}_{x_2} = 0,$$

where $\boldsymbol{U} = (h, u_1, u_2, Q_{11}, Q_{12}, Q_{22})$ and A and B are 6×6 matrices. Let $\boldsymbol{\xi} = (\tau, \xi, \eta)$ be the normal vector to the characteristics. Then, the characteristic matrix $A(\boldsymbol{\xi}) = \tau I + \xi A + \eta B$ of system (1.13) has the form

$$\begin{pmatrix} \chi & \xi h & \eta h & 0 & 0 & 0 \\ \xi\Big(\dfrac{hQ_{11}}{4} + \mathrm{Fr}^{-2}\Big) + \dfrac{\eta h Q_{12}}{4} & \chi & 0 & \dfrac{\xi h^2}{12} & \dfrac{\eta h^2}{12} & 0 \\ \dfrac{\xi h Q_{12}}{4} + \eta\Big(\dfrac{hQ_{22}}{4} + \mathrm{Fr}^{-2}\Big) & 0 & \chi & 0 & \dfrac{\xi h^2}{12} & \dfrac{\eta h^2}{12} \\ 0 & 2\eta Q_{12} & -2\eta Q_{11} & \chi & 0 & 0 \\ 0 & \eta Q_{22} - \xi Q_{12} & -\eta Q_{12} + \xi Q_{11} & 0 & \chi & 0 \\ 0 & -2\xi Q_{22} & 2\xi Q_{12} & 0 & 0 & \chi \end{pmatrix}.$$

Here and below, u_1 and u_2 denote the averaged fluid velocity components (the bar in the notation is omitted); $\chi = \tau + u_1\xi + u_2\eta$.

A simple but bulky calculation yields the following expression for $\det A(\boldsymbol{\xi})$:

$$\det A(\boldsymbol{\xi}) = \chi^2(\chi^2 - (h^2/12)(Q_{11}\xi^2 + 2Q_{12}\xi\eta + Q_{22}\eta^2))$$

$$\times (\chi^2 - (h^2/4)(Q_{11}\xi^2 + 2Q_{12}\xi\eta + Q_{22}\eta^2) - \mathrm{Fr}^{-2}\, h(\xi^2 + \eta^2)). \tag{2.1}$$

We specify the characteristic surface by the equation $H(t, x_1, x_2) = 0$. Then, to obtain the differential equations of the characteristics, we need to replace the vector (τ, ξ, η) in (2.1) by the vector (H_t, H_{x_1}, H_{x_2}) and equate $\det A(H_t, H_{x_1}, H_{x_2})$ to zero. As a result, we obtain a set of contact characteristics (the corresponding characteristic root has multiplicity two)

$$H_t + u_1 H_{x_1} + u_2 H_{x_2} = 0 \tag{2.2}$$

and four additional characteristic families

$$H_t + u_1 H_{x_1} + u_2 H_{x_2} = \pm\sqrt{(h^2/12)(Q_{11}H_{x_1}^2 + 2Q_{12}H_{x_1}H_{x_2} + Q_{22}H_{x_2}^2)}\,,$$

$$H_t + u_1 H_{x_1} + u_2 H_{x_2} = \pm\sqrt{(h/\mathrm{Fr}^2)(H_{x_1}^2 + H_{x_2}^2) + (h^2/4)(Q_{11}H_{x_1}^2 + 2Q_{12}H_{x_1}H_{x_2} + Q_{22}H_{x_2}^2)}\,.$$

The nonnegativeness of the quadratic form

$$Q_{11}H_{x_1}^2 + 2Q_{12}H_{x_1}H_{x_2} + Q_{22}H_{x_2}^2$$

is provided by inequality (1.14). Hence, if inequality (1.14) is satisfied, system (1.13) is hyperbolic. We note that if the terms containing Q_{ij} in the previous formulas are ignored, the formulas will contain only the contact characteristics (2.2) and analogs of the sonic characteristics of gas dynamics:

$$H_t + u_1 H_{x_1} + u_2 H_{x_2} = \pm(\sqrt{h}/\mathrm{Fr})\sqrt{H_{x_1}^2 + H_{x_2}^2}\,.$$

3. One-Dimensional Motion. In the case of one-dimensional motion, Eqs. (1.8) and (1.9) are written as

$$h_t + (hu)_x = 0, \quad (hu)_t + (hu^2)_x + (\mathrm{Fr}^{-2}\, h^2/2 + P)_x = 0, \quad P_t + uP_x + 3Pu_x = 0. \tag{3.1}$$

Performing the substitution of the sought function

$$P = \omega^2 h^3/12$$

(the quantity ω has the meaning of the average vorticity), we transform the last equation of system (3.1) to the form

$$(\omega^2)_t + u(\omega^2)_x = 0.$$

From this equation, it follows that ω^2 is conserved along the particle trajectories, as is the case for entropy in gas dynamics. System (3.1) implies the energy conservation law

$$\frac{\partial}{\partial t}\Big(\frac{hu^2}{2} + \frac{\omega^2 h^3}{24} + \frac{\mathrm{Fr}^{-2}\, h^2}{2}\Big) + \Big(\frac{hu^3}{2} + \frac{1}{8}\omega^2 h^3 u + \mathrm{Fr}^{-2}\, h^2 u\Big)_x = 0.$$

We define the internal energy of the "gas" e and the pressure $\tilde p$ by the formulae

$$e = \omega^2 h^2/24 + \mathrm{Fr}^{-2}\, h/2, \qquad \tilde p = \mathrm{Fr}^{-2}\, h/2 + \omega^2 h^3/12.$$

We calculate

$$de + \tilde p\, d\Big(\frac{1}{h}\Big) = \frac{h^2}{24}\, d\omega.$$

Determining the temperature of the "gas" $T = \omega^2 h^2/24$ and integrating the basic thermodynamic identity

$$ds = \frac{1}{T}\Big(de + \tilde p\, d\Big(\frac{1}{h}\Big)\Big) = \frac{d\omega^2}{\omega^2},$$

we find the entropy of the "gas":

$$s = \ln(\omega^2). \tag{3.2}$$

Thus, in the one-dimensional case, system (1.13) reduces to the equations of nonisentropic gas dynamics:

$$h_t + (hu)_x = 0, \qquad (hu)_t + (hu^2)_x + \tilde{p}_x = 0,$$
$$(h(u^2/2 + e))_t + (hu(u^2/2 + e + \tilde{p}/h))_x = 0. \tag{3.3}$$

The equations of state for the "gas" are written as

$$\tilde{p}(h, s) = \mathrm{Fr}^{-2}\, h^2/2 + e^s h^3/12, \qquad e(h, s) = \mathrm{Fr}^{-2}\, h/2 + e^s h^2/24,$$

where h plays the role of the density of the "gas" and the entropy s is related to the average vorticity by Eq. (3.2).

We note that fluid flows with constant average vorticity ($\omega = \text{const}$) correspond to isentropic gas flows, and the classical shallow-water equations are derived by passing to the limit $s \to -\infty$ in system (3.3).

The extension of the gas-dynamic analogy to vortical fluid flows presented here allows the well-known classes of solutions of the equations of one-dimensional nonisentropic gas dynamics to be used for an approximate description of free-boundary fluid flows. The obtained model of three-dimensional vortex free-boundary flows requires further investigation.

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